

AN ALEXANDER-TYPE DUALITY FOR VALUATIONS

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ABSTRACT. We prove an Alexander-type duality for valuations for certain subcomplexes in the boundary of polyhedra. These strengthen and simplify results of Stanley (1974) and Miller-Reiner (2005). We give a generalization of Brion's theorem for this relative situation and we discuss the topology of the possible subcomplexes for which the duality relation holds.

1. INTRODUCTION

Let $P \subset \mathbb{R}^d$ be a convex polytope with vertices in \mathbb{Z}^d and let $q \in \mathbb{R}^d$. Viewing q as a light source, let $B \subseteq \partial P$ be the collection of points in the boundary of P visible from q – the *bright side* of P . That is, B is the set of points $p \in \partial P$ for which the open segment (q, p) does not meet the relative interior of P . Sticking to these figurative terms, let D be the closure of the set of *dark points* $\partial P \setminus B$. Stanley [14] showed that for integral $n \geq 1$ the function

$$E_{P,B}(n) := |n \cdot (P \setminus B) \cap \mathbb{Z}^d|$$

is the restriction of a univariate polynomial (and, by abuse of notation, identified with that polynomial), and that

$$(1) \quad (-1)^{\dim P} E_{P,B}(-n) = |n \cdot (P \setminus D) \cap \mathbb{Z}^d| \quad \text{for all } n \geq 1.$$

By choosing $q \in \text{relint } P$, we have that $(B, D) = (\emptyset, \partial P)$ and (1) reduces to the well-known Ehrhart-Macdonald reciprocity [8]; see [2] for details. The set $B \subseteq \partial P$ is a particular case of what Ehrhart [5, 6] calls a *reciprocal domain*, that is, a domain for which (1) holds.

For a subset $S \subset \mathbb{R}^{d+1}$, the *lattice point enumerator* of S is the multivariate Laurent series

$$F_S(\mathbf{x}) := \sum_{a \in S \cap \mathbb{Z}^{d+1}} \mathbf{x}^a$$

where $\mathbf{x}^a = x_1^{a_1} x_2^{a_2} \cdots x_{d+1}^{a_{d+1}}$. If we associate to P the pointed cone $C(P) := \text{cone}(P \times \{1\}) \subset \mathbb{R}^{d+1}$, then $F_{C(P)}(\mathbf{x})$ records the individual lattice points $(a, n) \in \mathbb{Z}^{d+1}$ for which $a \in nP$. Stanley [14, Prop. 8.3] actually proved the stronger result that

$$(2) \quad (-1)^{\dim P} F_{C(P \setminus B)}\left(\frac{1}{\mathbf{x}}\right) = F_{C(P \setminus D)}(\mathbf{x}).$$

where $\frac{1}{\mathbf{x}} = (\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{d+1}})$.

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The relation (2) holds for general rational pointed polyhedral cones C but not for arbitrary subsets in the boundary of C . To see this, we can choose B as two non-adjacent triangles in the boundary of a 3-dimensional pyramid; one can check that B is not a reciprocal domain. The question which subsets in the boundary of C are reciprocal domains was investigated by Miller and Reiner [10]. They showed that the conditions giving rise to reciprocal domains are topological rather than geometric in nature. Let $C \subset \mathbb{R}^{d+1}$ be a rational, pointed polyhedral cone and let Δ be a full-dimensional subcomplex of the boundary of C , i.e. Δ is a polyhedral complex induced by a collection of facets of C . Let Δ' be the subcomplex generated by the facets $F \notin \Delta$. Their result is

Theorem 1.1 ([10, Thm. 1]). *If Δ is a Cohen-Macaulay complex, then*

$$(3) \quad (-1)^{d+1} F_{C \setminus |\Delta|} \left(\frac{1}{\mathbf{x}} \right) = F_{C \setminus |\Delta'|}(\mathbf{x}).$$

The proof of Theorem 1.1 in [10] is given in terms of combinatorial commutative algebra and relies on a connection between lattice point enumerators and Hilbert series of \mathbb{Z}^d -graded modules.

In this paper we give a simple proof of (1) and (3) that generalizes to a broader class of geometric objects and to valuations other than counting lattice points (see Theorem 3.1). Our proof relies on basic facts from topological combinatorics and, as a byproduct, gives a slightly more general class of complexes for which (1) holds. Like Theorem 1.1, our results are reminiscent of Alexander duality and we will emphasize this relation throughout.

The paper is organized as follows. In Section 2, we recall the notions of Λ -polytopes and valuations as well as (weakly) Cohen-Macaulay complexes. In Section 3 we state and prove an Alexander-duality type relation which contains Thm. 1.1 as a special case. In Section 4, we give a relative version of Brion's theorem which is interesting in its own right and highlights the role played by weakly Cohen-Macaulay complexes. In Section 5 we focus on the topology of full-dimensional (weakly) Cohen-Macaulay complexes in the boundary of spheres. The bright side B of P is homeomorphic to a ball of dimension $\dim P - 1$ and thus Cohen-Macaulay. A natural question, which was answered affirmatively in [10], is if there exist full-dimensional Cohen-Macaulay complexes in the boundary of polytopes that are not balls. We will extend this result and we discuss possibly counterintuitive instances for which (1) and (3) apply.

2. Λ -POLYTOPES, VALUATIONS, AND WEAKLY COHEN-MACAULAY COMPLEXES

We start by setting the stage for the use of more general geometric objects and valuations, following McMullen [9]. Throughout, let $\Lambda \subset \mathbb{R}^d$ be a fixed, full-dimensional discrete lattice or a vector space over some subfield of \mathbb{R} . We denote by $\mathcal{P} = \mathcal{P}(\Lambda)$ the collection of polytopes in \mathbb{R}^d with vertices in Λ . A Λ -valuation is a map φ from \mathcal{P} into some abelian group such that

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

whenever $P \cup Q \in \mathcal{P}$ (and hence $P \cap Q \in \mathcal{P}$) and such that $\varphi(t + P) = \varphi(P)$ for all $t \in \Lambda$. We can extend φ to *half-open* polytopes as follows. If $B \subset \partial P$ is the union of facets F_1, F_2, \dots, F_m of P , then

$$\varphi(P \setminus B) := \sum_{J \subseteq [m]} (-1)^{|J|} \varphi(F_J)$$

where $F_J := \bigcap \{F_j : j \in J\}$. In particular, if $B = \partial P$, we get

$$(4) \quad \varphi(\text{relint } P) = \sum_{F \subseteq P} (-1)^{\dim P - \dim F} \varphi(F)$$

where the sum is over all non-empty faces F of P . The following is the basis for our considerations.

Theorem 2.1 ([9]). *If φ is a Λ -valuation, then for all $n \in \mathbb{Z}_{\geq 0}$*

$$\varphi_P(n) := \varphi(nP)$$

agrees with a univariate polynomial of degree $\leq \dim P$ and

$$(-1)^{\dim P} \varphi_P(-1) = \varphi(\text{relint}(-P)).$$

A Λ -*complex* is a polyhedral complex \mathcal{K} such that every face is a Λ -polytope. The complex is *pure* if all inclusion-maximal faces have the same dimension. For example, the collection of proper faces of a Λ -polytope P is a pure Λ -complex, called the *boundary complex* $\mathcal{B}(P)$. The underlying set of \mathcal{K} is denoted by $|\mathcal{K}|$ and, since this is the disjoint union of relatively open polytopes, we can extend φ to Λ -complexes by setting

$$\varphi(|\mathcal{K}|) := \sum_{F \in \mathcal{K}} \varphi(\text{relint } F)$$

For a given face F in a polyhedral complex K , the *link* of F in K is the polyhedral subcomplex

$$\text{lk}_{\mathcal{K}}(F) = \{G \in \mathcal{K} : G \cap F = \emptyset, G \cup F \subseteq H \in \mathcal{K}\}.$$

For a subcomplex $\Delta \subset \mathcal{K}$, a face $F \in \Delta$ is an *interior face* of Δ if $\text{lk}_{\mathcal{K}}(F) \subset \Delta$ and a *boundary face* otherwise. The boundary of Δ is the subcomplex $\partial\Delta$ of all boundary faces. Note that for $F \notin \Delta$, we have $\text{lk}_{\Delta}(F) = \emptyset \neq \{\emptyset\}$ with reduced Euler characteristic $\tilde{\chi}(\emptyset) = 0$.

A pure complex \mathcal{K} is *weakly Cohen-Macaulay* if

$$\tilde{H}_i(\text{lk}_{\mathcal{K}}(F)) = 0 \quad \text{for all } 0 \leq i < \dim \text{lk}_{\mathcal{K}}(F).$$

for all non-empty faces $F \in \mathcal{K}$. Thus \mathcal{K} is Cohen-Macaulay if additionally $\tilde{H}_i(\mathcal{K}) = 0$ for all $0 \leq i < \dim \mathcal{K}$. This is a stronger condition as, for instance, weakly Cohen-Macaulay complexes are not necessarily connected. Since $G \subseteq F$ implies $\text{lk}_{\mathcal{K}}(F) \subseteq \text{lk}_{\mathcal{K}}(G)$, we get that K is weakly Cohen-Macaulay if and only if every vertex link of K is Cohen-Macaulay. Munkres [12] proved that Cohen-Macaulayness of a complex K is a topological property of the underlying pointset $|K|$ and hence K is weakly Cohen-Macaulay if $\tilde{H}_i(|K|, |K| \setminus p)$ vanishes for $i < \dim K$. Note that what we define is the notion of (weakly) \mathbb{Z} -CM complexes as our ring of coefficients is \mathbb{Z} throughout (cf. [3, Sect. 11]); however, most of our results hold for general rings of coefficients.

Finally, a pure Λ -complex \mathcal{K} of dimension d is a *homology manifold*, if for every face F of \mathcal{K} , the reduced homology of $\text{lk}_{\mathcal{K}}(F)$ is identically zero or if

$$\tilde{H}_*(\text{lk}_{\mathcal{K}}(F)) \cong \tilde{H}_*(S^{d - \dim F + 1}).$$

In particular, if $|\mathcal{K}|$ is a manifold, then \mathcal{K} is a homology manifold, and every homology manifold is weakly CM.

3. AN ALEXANDER-TYPE DUALITY

In this section we prove Alexander-type duality relations for Λ -valuations that relates complementary complexes Δ and Δ' inside Λ -complexes.

Theorem 3.1 (Alexander-type duality for valuations). *Let \mathcal{K} be a d -dimensional Λ -complex such that \mathcal{K} is a homology manifold with boundary and let $B \subset \partial\mathcal{K}$ be a full-dimensional, weakly Cohen-Macaulay subcomplex. Let D be the closure of $\partial\mathcal{K} \setminus B$. If φ is a Λ -valuation, then for all $n \geq 1$*

$$(-1)^d \varphi_{|\mathcal{K}| \setminus |B|}(-n) = \varphi_{-(|\mathcal{K}| \setminus |D|)}(n)$$

and

$$(-1)^d \varphi_{|\mathcal{K}| \setminus |B|}(0) = \varphi(\{0\})(\tilde{\chi}(\mathcal{K}) - \tilde{\chi}(B)) = \varphi_{-(|\mathcal{K}| \setminus |D|)}(0).$$

For the proof of the theorem we need to relate the combinatorics of inclusion-exclusion for the valuation φ to the topology of Δ . The main observation, captured in the following lemma, is that weakly Cohen-Macaulay $(d-1)$ -complexes which are embedded into the boundary of a d -dimensional homology manifold are rather restricted.

Lemma 3.2. *Let \mathcal{R} be a $(d-1)$ -dimensional homology manifold without boundary and let $\Delta \subseteq \mathcal{R}$ be a pure, weakly Cohen-Macaulay subcomplex of full dimension $d-1$. Then for every $\emptyset \neq F \in \Delta$*

$$\tilde{H}_k(\text{lk}_\Delta(F)) = \begin{cases} \mathbb{Z}, & \text{if } F \text{ is an interior face of dimension } d-k-2, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

In other words, a full-dimensional, weakly Cohen-Macaulay subcomplex of a homology manifold is again a homology manifold.

Proof. The link $\text{lk}_\Delta(F)$ is a subcomplex of $L = \text{lk}_\mathcal{R}(F)$, which has the homology of a k -sphere. Thus, if F is an interior face of Δ , then $\text{lk}_\Delta(F) = L$ and $\tilde{H}_*(\text{lk}_\Delta(F)) = \tilde{H}_*(S^k)$.

If $\text{lk}_\Delta(F) \subsetneq L$ is a proper subcomplex, it is sufficient to show that $\tilde{H}_k(\text{lk}_\Delta(F)) = 0$ for $k = \dim \text{lk}_\Delta(F)$, as Δ is weakly Cohen-Macaulay. For this observe that $|L| \setminus |\text{lk}_\Delta(F)|$ is non-empty. By Alexander duality for homology spheres [11, § 72], we get that

$$0 = \tilde{H}_{-1}(|L| \setminus |\text{lk}_\Delta(F)|) = \tilde{H}_k(\text{lk}_\Delta(F)).$$

Alternatively, it is sufficient to show that $\text{lk}_\Delta(F)$ is homotopic to a subcomplex of dimension $k-1$. To see this, note that $\text{lk}_\Delta(F)$ is a full-dimensional subcomplex of the k -dimensional homology manifold L . Thus, $\text{lk}_\Delta(F)$ has a *free face* and, using Whitehead's language of cellular collapses [15], $\text{lk}_\Delta(F)$ collapses to a subcomplex of its $(k-1)$ -skeleton. Since a collapse in particular provides a certificate for deformation retraction, this finishes the proof. \square

Proof of Theorem 3.1. As a subset of \mathbb{R}^d , $|\mathcal{K}|$ is partitioned by the relative interiors of faces $G \in \mathcal{K}$ and thus

$$\varphi_{|\mathcal{K}| \setminus |B|}(n) = \sum_{G \in \mathcal{K} \setminus B} \varphi_{\text{relint } G}(n),$$

For the case $n \neq 0$: as $\varphi_{|\mathcal{K}|}(n) = \varphi_{n|\mathcal{K}|}(1)$, is it is sufficient to prove the claim for $n = -1$. From Theorem 2.1 and (4), we get

$$\begin{aligned} (-1)^d \varphi_{|\mathcal{K} \setminus B|}(-1) &= \sum_{G \in \mathcal{K} \setminus B} (-1)^{d - \dim G} \varphi(-G) \\ &= \sum_{G \in \mathcal{K} \setminus B} (-1)^{d - \dim G} \sum_{\sigma \subseteq G \text{ face}} \varphi(\text{relint}(-\sigma)) \\ &= \sum_{\sigma \in \mathcal{K}} W_\sigma \varphi(\text{relint}(-\sigma)) \end{aligned}$$

where for a face $\sigma \in \mathcal{K}$

$$W_\sigma := (-1)^d \sum_{\sigma \subseteq G \in \mathcal{K} \setminus B} (-1)^{\dim G} = (-1)^{d - \dim G} (\tilde{\chi}(\text{lk}_\mathcal{K}(\sigma)) - \tilde{\chi}(\text{lk}_B(\sigma)))$$

It follows from Lemma 3.2, that $W_\sigma = 1$ if $\sigma \in \mathcal{K} \setminus D$ which proves the claim. The proof of the case $n = 0$ is analogous. \square

Since the boundary of every Λ -polytope is a sphere, we can extend the validity of (1) to general Λ -valuations.

Corollary 3.3. *Let $P \subset \mathbb{R}^d$ be a Λ -polytope. Let B be the underlying space of a full-dimensional, weakly CM subcomplex and let D be the closure of $\partial P \setminus B$. If φ is a Λ -valuation, then*

$$(-1)^{\dim P} \varphi_{P \setminus B}(-n) = \varphi_{-(P \setminus D)}(n) \text{ for all } n \neq 0.$$

This is indeed a generalization of (1), as $\varphi(S) = |S \cap \mathbb{Z}^d|$ is invariant under automorphisms of the lattice $\Lambda = \mathbb{Z}^d$. We give an example for a self-reciprocal domain, that is, $D = T(B) \subset \partial P$, where T is an automorphism of Λ with $T(P) = P$.

Example 3.4. Let $P = [0, 1]^4 = P_1 \times P_2$ be the 4-cube presented as the product of two squares $P_1 = P_2 = [0, 1]^2$. The boundary of the 4-cube contains a 2-dimensional torus $T = \partial P_1 \times \partial P_2$, which decomposes ∂P into two solid tori $S_1 = P_1 \times \partial P_2$ and $S_2 = \partial P_1 \times P_2$. As these are 3-manifolds with boundary, both S_1 and S_2 are pure 3-dimensional weakly Cohen-Macaulay subcomplexes. The Ehrhart function for a k -cube is $E_{[0,1]^k}(n) = (n+1)^k$. Thus the relative Ehrhart function is

$$E_{P, S_1}(n) = (n+1)^4 - 4(n+1)^3 + 4(n+1)^2 = n^4 - 2n^2 + 1$$

$$\text{and } (-1)^4 E_{P, S_1}(-n) = E_{P, S_2}(n) = E_{P, S_1}(n). \quad \diamond$$

Towards a proof for Theorem 1.1, let us record the following general lemma. For a polyhedral cone C and a point $a \in C$, let $\sigma_a \subset C$ be the unique face with $a \in \text{relint } \sigma_a$.

Lemma 3.5. *Let $C \subset \mathbb{R}^{d+1}$ be a rational $(d+1)$ -dimensional cone and $\Delta \subset \mathcal{B}(C)$ an arbitrary subcomplex. Then*

$$(-1)^{d+1} F_{C \setminus |\Delta|} \left(\frac{1}{\mathbf{x}} \right) = F_{\text{relint } C}(\mathbf{x}) + \sum_{a \in |\Delta| \cap \mathbb{Z}^{d+1}} (-1)^{d - \dim \sigma_a} \tilde{\chi}(\text{lk}_\Delta(\sigma_a)) \mathbf{x}^a.$$

Notice that $\text{lk}_\Delta(\sigma) \subseteq \text{lk}_{\mathcal{B}(C)}(\sigma) \cong S^{d - \dim \sigma}$. Thus, the coefficient of \mathbf{x}^a in the equation above is the Euler characteristic of the Alexander dual of $|\text{lk}_\Delta(\sigma_a)| \subset S^{d - \dim \sigma_a}$.

Proof. From Ehrhart theory (cf. [14, Prop. 7.1]), we have for a rational cone G

$$(-1)^{\dim G} F_G\left(\frac{1}{\mathbf{x}}\right) = F_{\text{relint } G}(\mathbf{x}).$$

Thus, from

$$F_{C \setminus |\Delta|}(\mathbf{x}) = F_C(\mathbf{x}) - \sum_{G \in \Delta} F_{\text{relint } G}(\mathbf{x})$$

we obtain

$$(-1)^{d+1} F_{C \setminus |\Delta|}\left(\frac{1}{\mathbf{x}}\right) = F_{\text{relint } C}(\mathbf{x}) + \sum_{G \in \Delta} (-1)^{d-\dim G} F_G(\mathbf{x})$$

which shows that the right-hand side is supported on $\text{relint}(C) \cup |\Delta|$. Now for $a \in |\Delta| \cap \mathbb{Z}^{d+1}$, the coefficient of \mathbf{x}^a on the right-hand side is

$$(-1)^{d+1} \sum_{\sigma_a \subseteq G \in \Delta} (-1)^{\dim G} = (-1)^{d-\dim \sigma_a} \sum_{\bar{G} \in \text{lk}_\Delta(\sigma_a)} (-1)^{\dim \bar{G}} = (-1)^{d-\dim \sigma_a} \tilde{\chi}(\text{lk}_\Delta(\sigma_a)),$$

which proves the claim. \square

Proof of Theorem 1.1. If Δ is Cohen-Macaulay, then for every face $F \in \Delta$, the link $\text{lk}_\Delta(F)$ has the reduced Euler characteristic of a $(d-1-\dim F)$ -sphere if F is interior and the reduced Euler characteristic of a point otherwise. Together with Lemma 3.5 this gives us

$$(-1)^{d+1} F_{C \setminus |\Delta|}\left(\frac{1}{\mathbf{x}}\right) = F_{\text{relint } C}(\mathbf{x}) + \sum_{a \in (|\Delta| \setminus |\Delta'|) \cap \mathbb{Z}^{d+1}} \mathbf{x}^a \quad \square$$

4. A RELATIVE BRION THEOREM

In this section we give a version of Brion's theorem [4] (see also [1]) suitable in the presence of a *forbidden* subcomplex. To make our results more transparent, let us start with the classical Brion-Gram relation for polytopes and an interesting complementary version. For a subset $S \subseteq \mathbb{R}^d$, let us denote by $[S] : \mathbb{R}^d \rightarrow \{0, 1\}$ the indicator function. Note, that $[S \cap T] = [S] \cdot [T]$.

Let $C = \{x \in \mathbb{R}^{d+1} : \langle a_i, x \rangle \leq 0 \text{ for } i = 1, 2, \dots, m\}$ be a polyhedral cone. For a non-empty face $F \subseteq C$ let $I(F) = \{i \in [m] : \langle a_i, x \rangle = 0 \text{ for all } x \in F\}$ and define the *tangent cone* of C at F as

$$T_C(F) := \{x \in \mathbb{R}^d : \langle a_i, x \rangle \leq 0 \text{ for } i \in I(F)\}$$

Lemma 4.1. *Let $C \subset \mathbb{R}^{d+1}$ be a full-dimensional polyhedral cone. Then*

$$\sum_{\emptyset \neq F \subseteq C} (-1)^{\dim F} [T_C(F)] = (-1)^{d+1} [\text{int}(-C)]$$

Proof. If $p \in \text{int}(-C)$, then $p \in T_C(F)$ if and only if $F = C$. For $p \in \mathbb{R}^{d+1} \setminus \text{int}(-C)$, let $J = \{i \in [m] : \langle a_i, p \rangle \leq 0\}$. Then

$$C_J := \{x \in \mathbb{R}^{d+1} : \langle a_i, x \rangle \leq 0 \text{ for } i \in J\}$$

is the product of a linear space and a pointed polyhedral cone and thus has Euler characteristic $\chi(C_J) = 0$. Moreover, by sending the point $p \in C_J$ to infinity, the faces of C_J are exactly those faces $F \subseteq C$ for which $p \in T_C(F)$ and the left-hand side of the stated equation computes the Euler characteristic of C_J . \square

From that, we can deduce the usual Brianchon-Gram relation. If $P \subset \mathbb{R}^d$ is polytope and F a face, then the tangent cone of P at F is defined analogously as above and, equivalently, $T_P(F) = q_F + \text{cone}(P - q_F)$, where $q_F \in \text{relint } F$. In particular, we have $T_P(P) = \mathbb{R}^d$ and $T_P(\emptyset) = P$.

Corollary 4.2. *If $P \subset \mathbb{R}^d$ is a polytope, then*

$$[P] = \sum_{\emptyset \neq F \subseteq P} (-1)^{\dim F} [T_P(F)].$$

Proof. Let $C = C(P) \subset \mathbb{R}^{d+1}$ be cone associated to P . Let $H = \mathbb{R}^d \times \{1\}$. Then the $(k+1)$ -faces \hat{F} of C bijectively correspond to k -faces under $F = \hat{F} \cap H$. With the appropriate identifications, $[P] = [C] \cdot [H]$ and, in particular, $[T_P(F)] = [T_C(\hat{F})] \cdot [H]$. Since $H \cap \text{int}(-C) = \emptyset$, we get from Lemma 4.1

$$[P] = [H] \cdot [T_C(0)] = [H] \sum_{\{0\} \subsetneq \hat{F} \subseteq C} (-1)^{\dim \hat{F}-1} [T_C(\hat{F})] = \sum_{\emptyset \neq F \subseteq P} (-1)^{\dim F} [T_P(F)],$$

which proves the claim. \square

We also get an interesting complementary version as follows. For every face $F \subseteq P$, the tangent cone is of the form $T_P(F) = \text{aff}(F) + C_P(F)$ where $C_P(F)$ is the unique cone contained in $\text{aff}(F)^\perp$. Let us define the *inverted tangent cone* as $T_P^{-1}(F) = \text{aff}(F) - C_P(F)$.

Corollary 4.3. *Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope. Then*

$$(-1)^d [\text{relint}(P)] = \sum_{\emptyset \neq F \subseteq P} (-1)^{\dim F} [T_P^{-1}(F)].$$

Proof. Let $P = \{x : \langle a_i, x \rangle \leq b_i \text{ for } i \in [m]\}$. For a non-empty face $F \subseteq P$, the inverted tangent cone is given by

$$T_P^{-1}(F) = \{x \in \mathbb{R}^d : \langle a_i, x \rangle - b_i \geq 0 \text{ for } i \in I(F)\}.$$

Now consider $C = C(-P) = \{(x, t) : t \geq 0, \langle -a_i, x \rangle - b_i t \leq 0, i \in [m]\}$ and $H = \mathbb{R}^d \times \{-1\}$. Then, with appropriate identifications, $\text{relint}(-C) \cap H = \text{relint}(P)$ and $T_C(\hat{F}) \cap H = T_P^{-1}(F)$. Lemma 4.1 now yields the result. \square

For dealing with forbidden subcomplexes, we will also need the following relative versions of the two Brianchon-Gram relations. If $\Delta \subseteq \mathcal{B}(P)$ is a full-dimensional subcomplex of the boundary, then this induces a subcomplex $\Delta_F \subseteq \mathcal{B}(T_P(F))$ in the tangent cone of every face $F \subsetneq P$. This subcomplex is pure of dimension $d-1$ or empty. We write $T_{P,\Delta}(F) = T_P(F) \setminus |\Delta_F|$ for the tangent cone minus the faces induced by Δ , and $T_{P,\Delta}^{-1}(F)$ for the analogously defined relative inverted tangent cone.

Lemma 4.4. *Let $P \subset \mathbb{R}^d$ be a d -polytope and $\Delta \subseteq \mathcal{B}(P)$ a full-dimensional subcomplex. Let $\Delta' \subseteq \mathcal{B}(P)$ be the subcomplex spanned by the facets not contained in Δ . Then*

$$[P \setminus |\Delta|] = \sum_F (-1)^{\dim F} [T_{P,\Delta}(F)]$$

and

$$(-1)^{\dim P} [P \setminus |\Delta'|] = \sum_F (-1)^{\dim F} [T_{P,\Delta}^{-1}(F)]$$

where the sums are over all non-empty faces $F \subseteq P$.

Proof. We prove only the first statement as the proof of the second relation is analogous. Let $p \in \mathbb{R}^d$ be an arbitrary point. If p is not contained in the affine span of any face of Δ , then $[T_{P,\Delta}(F)](p) = [T_P(F)](p)$ for all non-empty faces $F \subseteq P$ and the identity is Corollary 4.2. Thus, suppose that p is contained in some hyperplane spanned by a facet in Δ .

If $p \in P$, then the unique face $F \subseteq P$ containing p in the relative interior is a face of Δ . In this case $p \in T_{P,\Delta}(G)$ if and only if G and F are contained in a common face of Δ . That is, if D is contained in the *closed star* $\text{st}_\Delta(F) := \{G \in \Delta : F \cup G \subseteq K \in \Delta\}$ of F in Δ . The right-hand side of the stated equation evaluated at p can be written as

$$\sum_{F \in \mathcal{B}(P) \setminus \{\emptyset\}} (-1)^{\dim F} - \sum_{D \in \text{st}_\Delta(F) \setminus \{\emptyset\}} (-1)^{\dim D}.$$

This is the difference of the unreduced Euler characteristics of two contractible complexes and therefore $0 = 1 - 1$.

If $p \in \mathbb{R}^d \setminus P$, let $F_1, \dots, F_k \subseteq P$ be the $(d-1)$ -dimensional faces of Δ for which p is contained in the affine hyperplane $H_i := \text{aff}(F_i)$ spanned by F_i . We have to show that

$$(5) \quad \sum \{(-1)^{\dim G} : p \in T_P(G) \text{ and } G \subseteq F_i \text{ for some } i = 1, \dots, k\} = 0,$$

as this is the collection of terms missing from the usual Brianchon-Gram. For $I \subseteq [k]$, let $F_I = \cap_{i \in I} F_i$ and define

$$s_I := \sum \{(-1)^{\dim G} : p \in T_P(G) \text{ and } G \subseteq F_I\}.$$

We can rewrite the left-hand side of (5) as

$$\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|-1} s_I.$$

But for a fixed I , we have that s_I is equal to the left-hand side of the Brianchon-Gram relation applied to F_I and a point $p \notin F_I$ inside $\text{aff}(F_I)$. Thus $s_I = 0$. \square

We can now state our generalization of Brion's theorem.

Theorem 4.5 (Relative Brion's theorem). *Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with vertices $v_1, v_2, \dots, v_n \in \mathbb{Z}^d$. Let $\Delta \subseteq \mathcal{B}(P)$ a pure and d -dimensional weakly Cohen-Macaulay subcomplex and let $\Delta' \subseteq \mathcal{B}(P)$ be the subcomplex generated by the facets of P not contained in Δ . Then*

$$\mathbf{F}_{P \setminus |\Delta|}(\mathbf{x}) = \mathbf{F}_{T_{P,\Delta}(v_1)}(\mathbf{x}) + \mathbf{F}_{T_{P,\Delta}(v_2)}(\mathbf{x}) + \dots + \mathbf{F}_{T_{P,\Delta}(v_n)}(\mathbf{x})$$

and

$$(-1)^d \mathbf{F}_{P \setminus |\Delta|}(\frac{1}{\mathbf{x}}) = \mathbf{F}_{-(P \setminus |\Delta'|)}(\mathbf{x}).$$

Proof. The first statement follows from the same consideration as in [1]: Observe that for $S \subset \mathbb{R}^d$, we have $F_S(x) = \sum_{a \in \mathbb{Z}^d} [S](a) \mathbf{x}^a$ and from Lemma 4.4 we get

$$F_{P \setminus |\Delta|}(\mathbf{x}) = \sum_F (-1)^{\dim F} F_{T_{P,\Delta}(F)}(\mathbf{x})$$

where the sum is over all non-empty faces $F \subseteq P$. Now if F is not a vertex, the relative tangent cone $T_{P,\Delta}(F)$ is not pointed, that is, $t + T_{P,\Delta}(F) = T_{P,\Delta}(F)$ for some $t \neq 0$. On the level of lattice point enumerators, this means $\mathbf{x}^t F_{T_{P,\Delta}(F)}(\mathbf{x}) = F_{T_{P,\Delta}(F)}(\mathbf{x})$ and thus $F_{T_{P,\Delta}(F)}(\mathbf{x}) = 0$. This proves the first statement.

By the same token, we get from Lemma 4.4

$$(-1)^d F_{P \setminus |\Delta'|}(\mathbf{x}) = \sum_F (-1)^{\dim F} F_{T_{P,\Delta}^{-1}(F)}(\mathbf{x})$$

and thus

$$(-1)^d F_{P \setminus |\Delta'|}(\mathbf{x}) = \sum_{i=1}^n F_{T_{P,\Delta}^{-1}(v_i)}(\mathbf{x})$$

Let us write $T_{P,\Delta}(v_i) = v_i + C_i \setminus |\Delta_{v_i}|$ where C_i is a rational polyhedral cone and $\Delta_i = \Delta_{v_i}$ is the subcomplex induced by Δ . In particular, $F_{T_{P,\Delta}(v_i)}(\mathbf{x}) = \mathbf{x}^{v_i} F_{C_i \setminus |\Delta_i|}(\mathbf{x})$. Since Δ is weakly Cohen-Macaulay, we have that Δ_i is Cohen-Macaulay and by Theorem 1.1

$$(-1)^d F_{T_{P,\Delta}(v_i)}(\frac{1}{\mathbf{x}}) = \mathbf{x}^{-v_i} F_{C_i \setminus |\Delta'_i|}(\mathbf{x}) = F_{T_{-P,-\Delta'}^{-1}(-v_i)}(\mathbf{x})$$

For the finishing touch, we calculate

$$F_{P \setminus |\Delta|}(\frac{1}{\mathbf{x}}) = \sum_{i=1}^n F_{T_{P,\Delta}(v_i)}(\frac{1}{\mathbf{x}}) = \sum_{i=1}^n (-1)^d F_{T_{-P,-\Delta'}^{-1}(-v_i)}(\mathbf{x}) = (-1)^d F_{-P \setminus |\Delta'|}(\mathbf{x}) \quad \square$$

5. TOPOLOGY OF RECIPROCAL DOMAINS

Theorem 1.1 and Theorem 3.1 apply to full-dimensional (weakly) Cohen-Macaulay complexes in the boundaries of polytopes. In this section we discuss what forms these complexes can take. In [10], Miller and Reiner gave an example of a full-dimensional Cohen-Macaulay subcomplex in the boundary of a polytope that is not contractible and hence not a ball; they argued that, for instance, the Mazur manifold can occur. The purpose of this section is to generalize this remark. We refer to [13] for the basic notions of PL topology.

Theorem 5.1. *Let B be any PL manifold of dimension $d \geq 5$ such that*

- (a) *The natural inclusion $\pi_1(\partial B) \hookrightarrow \pi_1(B)$ is surjective, and*
- (b) *B is homologically trivial, i.e., $\tilde{H}_*(B) = \tilde{H}_*(B_d)$.*

Then there exists a $(d+1)$ -polytope P and a subcomplex $\tilde{B} \subseteq \mathcal{B}(P)$ such that \tilde{B} is PL-homeomorphic to B . In particular, \tilde{B} is a full-dimensional weakly Cohen-Macaulay subcomplex of ∂P .

Any homology manifold B satisfying assumptions (a) and (b) is a *homology ball*.

Proof. By [7, Thm. 3], there is a contractible PL manifold M for which ∂M is PL homeomorphic to ∂B . Then the gluing of M and B along their boundaries is a PL-sphere S , since it is PL (because B and M are PL), simply connected (by property (a) of B and the fact that M is contractible) and has the homology of a sphere (since both M and B have the homology of a sphere); consequently, it is a PL sphere by the generalized Poincaré conjecture [16]. In particular, we have that there exists a subdivision S' of S that is combinatorially equivalent to the boundary complex S'' of a $(d+1)$ -polytope P . The subcomplex of S'' corresponding to B is the desired complex \tilde{B} . \square

Corollary 5.2. *Every contractible PL d -manifold B , $d \geq 5$ can be realized, up to PL homeomorphism, as a full-dimensional weakly Cohen-Macaulay subcomplex in the boundary of a $(d+1)$ -polytope.*

This suggests that every PL manifold satisfying (a) and (b) of Theorem 5.1 is contractible. This is not the case:

Example 5.3. Let S denote a PL homology sphere that is not S^d , such as Poincaré's homology sphere, and let Δ denote any facet of S . Then $B := (S - \Delta) \times [0, 1]$ is a homology ball, but homotopy equivalent to $S - \Delta$, which has $\pi_1(S) = \pi_1(S - \Delta) \neq 0$ and is consequently not contractible.

Theorem 3.1 applies more generally to subcomplexes in the boundary of homology manifolds; in this case, we are surprisingly flexible:

Theorem 5.4. *Let M denote any homology manifold with vanishing reduced homology. Then there exists a homology ball that contains M as a full-dimensional subcomplex of its boundary.*

Proof. Let $D(M, \partial M)$ denote the double of M (that is, the result of gluing two manifolds PL homeomorphic to M along their isomorphic boundaries). By excision, the complex $D(M, \partial M)$ is a homology manifold without boundary which is homologically equivalent to a sphere. Thus, the cone over $D(M, \partial M)$ is a homology ball, as desired. \square

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